Operationally Invariant Information in Quantum Measurements

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A new measure of information in quantum mechanics is proposed which takes into account that for quantum systems the only features known before an experiment is performed are the probabilities for various events to occur. The sum of the individual measures of information for mutually complementary observations is invariant under the choice of the particular set of complementary observations and conserved if there is no information exchange with an environment. That operational quantum information invariant results in $k$ bits of information for a system consisting of $k$ qubits.

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In any individual quantum measurement with discrete variables a number of different outcomes are possible, for example, in a spin-1/2 measurement the individual outcomes “spin up” and “spin down.” We define a new measure of information for an individual quantum measurement based on the fact that the only features defined before the measurement is performed are the specific probabilities for all possible individual outcomes.

The observer is free to choose different experiments which might even completely exclude each other, for example, measurements of orthogonal components of spin. This quantum complementarity of variables occurs when the corresponding operators do not commute. One quantity, for example, the $z$ component of spin, might be well defined at the expense of maximal uncertainty about the other orthogonal components. We define the total information content in a quantum system to be the sum over all individual measures for a complete set of mutually complementary experiments.

The experimentalist may decide to measure a different set of complementary variables thus gaining certainty about one or more variables at the expense of losing certainty about other(s). In the case of spin these could be the projections along rotated directions, for example, where the uncertainty in one component is reduced but in another component it is increased correspondingly.

Intuitively one expects that the total uncertainty or, equivalently, the total information carried by the system is invariant under such transformation from one complete set of complementary variables to another. We show that the total information defined according to our new measure has exactly that invariance property. Also it is conserved in time if there is no information exchange with an environment.

We find that the total information of a system results in $k$ bits of information for a system consisting of $k$ qubits. For a composite system, maximal entanglement results if the total information carried by the system is exhausted in specifying joint properties, with no individual qubit carrying any information on its own. Our results we interpret as implying that information is the most fundamental notion in quantum mechanics.

Every reasonably well-designed experiment tests some proposition. Knowledge of the state of a quantum system permits the prediction of individual outcomes with certainty only for that limited class of experiments which have definite outcomes, a situation where the corresponding propositions have definite truth values. From theorems like Kochen-Specker [1] we know that in quantum mechanics it is not possible, not even in principle, to assign definite noncontextual truth values to all conceivable propositions. About indefinite propositions we can make only probabilistic predictions.

Consider an experimental arrangement with $n$ possible outcomes. Knowing the probabilities $p_j = (p_1,\ldots,p_j,\ldots,p_n)$ for the outcomes all an experimenter can do is to guess how many times a specific outcome will occur. In making his prediction he has only a limited number of systems to work with. Then, because of the statistical fluctuations associated with any finite number of experimental trials, the number $n_j$ of occurrences of a specific outcome $j$ in future $N$ repetitions of the experiment is not precisely predictable. Rather, the experimenter’s uncertainty (mean-square deviation), or lack of information, in the value $n_j$ is [2]

$$\sigma_j^2 = p_j(1 - p_j)N .$$

This implies that for a sufficiently large number $N$ of experimental trials the confidence interval is given as $(p_jN - \sigma_j, p_jN + \sigma_j)$. Therefore, if we just plan to perform the experiment $N$ times, we know in advance, before the experiments are performed and their outcomes become known, that the number $n_j$ of future occurrences of the outcome $j$ will be found with probability 68% within the confidence interval.

Notice that the experimenter’s lack of information (1) is proportional to the number of trials. This important property guarantees that each individual performance of the experiment contributes the same amount of information, no matter how many times the experiment has already
normalization is

or

This is the lack of information about the outcome \( j \) with respect to a single future experimental trial. In this view we suggest to define the whole lack of information regarding all \( n \) possible experimental outcomes as

\[
U(\vec{p}) = \sum_{j=1}^{n} U(p_j) = \sum_{j=1}^{n} p_j(1 - p_j) = 1 - \sum_{j} p_j^2.
\] (3)

The uncertainty is minimal if one probability is equal to one and it is maximal if all probabilities are equal.

This suggests that the knowledge, or information, with respect to a single future experimental trial an experimentalist possesses before the experiment is performed is a complement of \( U(\vec{p}) \) and, furthermore, that it is a function of a sum of the squares of probabilities. A first ansatz therefore would be

\[
I(\vec{p}) = 1 - U(\vec{p}) = \sum_{i=1}^{n} p_i^2.
\] (4)

A set of propositions associated with certain quantum-mechanical experiments is mutually complementary if complete knowledge of the truth value of any one of the propositions implies maximal uncertainty about the truth values of the others. Such a complete set of propositions for a spin-1/2 particle can be for example the following: (1) “The spin along the \( x \) axis is up,” (2) “the spin along the \( z \) axis is up,” and (3) “the spin along the \( y \) axis is up” \([6]\).

Another example for complementarity is quantum interference. Consider an experiment with an idealized Mach-Zehnder type of interferometer (Fig. 1). Suppose that for a specific phase shift \( \phi \) between the two beams inside the interferometer (Fig. 1a), the particle will exit with certainty towards the upper (lower) detector behind the beam splitter. In this case we have complete knowledge of the beam the particle will be found in behind the beam splitter at the expense of the fact that we have absolutely no knowledge which path the particle took inside the interferometer. The state of the particle is then represented by the truth value (true or false) of the proposition (1): “The particle takes the outgoing path towards the upper detector in presence of the phase shift \( \phi \).”

In contrast, if we know which path the particle took through the interferometer (Fig. 1b) no interference results and hence it is completely uncertain which outgoing path the particle will take. The state of the particle can now be specified by the truth value of the proposition (2): “The particle takes the upper path inside the interferometer.”

Knowing that spin-1/2 affords a model of the quantum mechanics of all two-state systems, i.e., qubits, we expect...
that there are always three mutually complementary propositions whenever binary alternatives are considered. Indeed, it can easily be shown that even without path information our knowledge of the beam the particle will be found in behind the beam splitter in Fig. 1a will be completely removed if we introduce an additional phase shift of $\pi/2$ between the two beams inside the interferometer. Then, in the new arrangement in Fig. 1c both outgoing beams will be equally probable.

Now, suppose that in the presence of a specific phase shift $\phi + \pi/2$ (Fig. 1c), the particle will exit with certainty towards the upper (lower) detector. The state of the system is now represented by the truth value of the proposition (3): “The particle takes the outgoing path towards the upper detector in presence of the phase shift $\phi + \pi/2$.” For a particle in that state we have complete knowledge of the outgoing beam the particle will take (Fig. 1c) at the expense of absolutely no knowledge about either the path inside the interferometer (Fig. 1b) or about the outgoing path in the arrangement in Fig. 1a.

Notice that we can label various sets of the three mutually complementary propositions by the value $\phi$ of the phase shift. The three propositions we found for the interferometer are formally equivalent to the complementary phase shift. The three propositions we found for the in-

ventory are formally equivalent to the complementary

Now for the total information carried by the composite system

$$I_{\text{total}} = \sum_{j=1}^{5} I_j(\tilde{p}^j) = \frac{2}{3} (4 \text{Tr} \rho^2 - 1).$$

Here, $\tilde{p}^j = (p^j_1, p^j_2, p^j_3, p^j_4)$ are the probabilities for the system in the state $\rho$ to give the four possible
combinations (true-true, true-false, false-true, and false-false) of the truth values for the pair \( j \) of propositions. This again is invariant under unitary transformations. Independence of physical parameters \( \phi_1 \) and \( \phi_2 \) implies that the total information of the composite system is invariant under the choice of the particular set of five mutually complementary pairs of propositions. Also the total information of the composite system is conserved in time if there is no information exchange between the composite system and an environment. We note that these results can be generalized to a composite system consisting of \( k \) qubits.

A composite 2-qubit system in a pure state carries 2 bits of information. That information contained in two propositions can be distributed over the two particles in various ways. It may be carried by the two particles individually, e.g., as the two-bit combination false-true of the truth values of the propositions given in (1). This is then represented by the product state \( |\psi_\text{prod}\rangle = |z-\rangle |z+\rangle \). The 2 bits of information are thus encoded in the two particles separately, one bit in each particle just as in classical physics. In that case there is no additional information represented jointly by the two systems.

Alternatively, 2 bits of information might all be carried by the two particles in a joint way, in the extreme with no individual particle carrying any information on its own. For example, this could be the two-bit combination true-false of the truth values of the propositions given in (4). Again, this is represented by the entangled state

\[
|\psi\rangle_{\text{ent}} = \frac{1}{\sqrt{2}} \left( |z+\rangle |x(\phi_2)+\rangle + |z-\rangle |x(\phi_2)-\rangle \right) = \frac{1}{\sqrt{2}} \left( |x(\phi_1)+\rangle |y(\phi_2)-\rangle - i |x(\phi_1)-\rangle |y(\phi_2)+\rangle \right),
\]

where, e.g., \( |x(\phi)+\rangle \) represents the eigenstate spin-up along a direction rotated by \( \phi_1 \) from \( x \). This Bell state does not contain any information about the individuals; all information is contained in joint properties. In fact, now there cannot be any information carried by the individuals because the two bits of information are exhausted by defining that maximally entangled state, and no further possibility exists to also encode information in individuals. This we see as a quantitative formulation of Schrödinger’s [10] idea that “if two separated bodies, each by itself known maximally, enter a situation in which they influence each other, and separate again, then... the knowledge remains maximal, but at its end, if the two bodies have again separated, it is not again split into a logical sum of knowledges about the individual bodies.”

For clarity we emphasize that our total information content of a quantum system is neither mathematically nor conceptually equivalent to von Neumann’s entropy. With the only exception for results of measurement in a basis decomposing the density matrix into a classical mixture when it can be considered as equivalent to Shannon’s information, the von Neumann entropy is just a measure of the purity of the given density matrix without explicit reference to information contained in individual measurements [4]. In contrast, our information content is purely operational and refers directly to experimental results of mutually complementary measurements, thus including also those for which the density matrix cannot be decomposed into a classical mixture. Our information content of the system can be viewed as equivalent to the sum of partial knowledge an experimentalist can have about mutually exclusive measurements without any further reference to the structure of the theory.

In the present paper we find an operational quantum information invariant that reflects the intrinsic symmetry of the underlying Hilbert space of the system. We interpret our result as implying that number of essential features of quantum mechanics might be based on the observation [11,12] that \( N \) most elementary systems represent the truth values of \( N \) propositions only. Since this is the only information quantum systems carry, a measurement associated with any other independent proposition must necessarily contain an element of randomness. This kind of randomness must then be irreducible, that is, it cannot be reduced to “hidden” properties of the system. Otherwise the system would carry more information.

Entanglement results from the fact that information could also be distributed in joint properties of a multiparticle system. In particular, maximal entanglement arises when the total information of a composite system is exhausted in specifying joint properties.

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[6] These statements we only mean to be statements about detector clicks in Stern-Gerlach type experiments.
[10] E. Schrödinger, Naturwissenschaften 23, 807 (1935); see also www.emr.hibu.no/lars/eng/cat